Sphere model calculations

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The calculations in this document relate to my article ‘A new model to explain the forces between moving charges’, published by *Physics Essays* in July 2018.

Hereafter I will refer to this article as the ‘sphere model article’.

A Classical calculations

Using the abbreviation

$$k = \frac{q_1\left(1-\frac{u^2}{c^2}\right)}{4\pi\varepsilon_o r^2 \left(1-\frac{u^2}{c^2}\,sin^2\,\alpha\right)^{\frac{3}{2}}} \quad \text{(A1)}$$

when a charge q_1 moves at the constant velocity $\mathbf{u}$ as shown in Fig.1 in the sphere model article, then at the origin of the coordinate system shown in Fig. 1 the electric field associated with that charge is

$$\mathbf{E} = k\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} \quad \text{(A2)}$$

and the magnetic field in the same location is

$$\mathbf{B} = \frac{\mathbf{u}\times\mathbf{E}}{c^2} = \frac{k}{c^2}\begin{pmatrix}u_x\\ u_y\\ u_z\end{pmatrix}\times\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} = \frac{k}{c^2}\begin{pmatrix}-u_z\sin\theta\\ u_z\cos\theta\\ u_x\sin\theta - u_y\cos\theta\end{pmatrix} \quad \text{(A3)}$$

For the components of $\mathbf{u}$, we have

$$u_x = u_{xy}\cos(\theta+\gamma) = u\cos\delta\,(\cos\theta\cos\gamma - \sin\theta\sin\gamma)$$

$$u_y = u_{xy}\sin(\theta+\gamma) = u\cos\delta\,(\sin\theta\cos\gamma + \cos\theta\sin\gamma)$$

$$u_z = u\sin\delta \qquad \text{(A4)}$$

Inserting (A4) in (A3), we obtain

$$\mathbf{B} = \frac{ku}{c^2}\begin{pmatrix}-\sin\delta\sin\theta\\ \sin\delta\cos\theta\\ -\cos\delta\sin^2\theta\sin\gamma - \cos\delta\cos^2\theta\sin\gamma\end{pmatrix} = \frac{ku}{c^2}\begin{pmatrix}-\sin\delta\sin\theta\\ \sin\delta\cos\theta\\ -\cos\delta\sin\gamma\end{pmatrix} \quad \text{(A5)}$$

According to the Lorentz force law, the force on q_2 in Σ is thus

$$\mathbf{F} = q_2(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = q_2 k\left[\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} + \frac{u}{c^2}\begin{pmatrix}v\\ 0\\ 0\end{pmatrix}\times\begin{pmatrix}-\sin\delta\sin\theta\\ \sin\delta\cos\theta\\ -\cos\delta\sin\gamma\end{pmatrix}\right] \quad \text{(A6)}$$

Substituting for *k* from (A1) and calculating the cross product, we obtain:

$$\mathbf{F} = \frac{q_1 q_2\left(1-\frac{u^2}{c^2}\right)}{4\pi\varepsilon_o r^2\left(1-\frac{u^2}{c^2}\,sin^2\,\alpha\right)^{\frac{3}{2}}}\begin{pmatrix}\cos\theta\\ \sin\theta + \frac{uv}{c^2}\sin\gamma\cos\delta\\ \frac{uv}{c^2}\sin\delta\cos\theta\end{pmatrix} \quad \text{(A7)}$$

To obtain the magnitude of the force on q_2 as measured by a co-moving spring balance, we need to make a relativistic force transformation to obtain the force in the rest frame of q_2:

$F'_x = F_x$, $F'_y = F_y/\sqrt{1 - v^2/c^2}$ and $F'_z = F_z/\sqrt{1 - v^2/c^2}$. We thus have

$$\mathbf{F}' = \frac{q_2 q_1 \left(1 - \frac{u^2}{c^2}\right)}{4\pi\varepsilon_o r^2 \left(1 - \frac{u^2}{c^2}\ sin^2\ \alpha\right)^{\frac{3}{2}} \sqrt{1 - \frac{v^2}{c^2}}} \begin{pmatrix} \cos\theta \sqrt{1 - \frac{v^2}{c^2}} \\ \sin\theta + \frac{uv}{c^2}\sin\gamma\cos\delta \\ \frac{uv}{c^2}\sin\delta\cos\theta \end{pmatrix} \quad \text{(A8)}$$

To calculate the magnitude of this force, we first determine

$$\begin{aligned}
\begin{pmatrix} \cos\theta \sqrt{1 - \frac{v^2}{c^2}} \\ \sin\theta + \frac{uv}{c^2}\sin\gamma\cos\delta \\ \frac{uv}{c^2}\sin\delta\cos\theta \end{pmatrix}^2
&= \cos^2\theta - \frac{v^2}{c^2}\cos^2\theta + \sin^2\theta \\
&+ 2\frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta + \frac{u^2v^2}{c^4}\sin^2\gamma\cos^2\delta + \frac{u^2v^2}{c^4}\cos^2\theta\ \sin^2\delta \\
&= 1 + 2\frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta + \frac{u^2v^2}{c^4}\sin^2\theta\sin^2\gamma\cos^2\delta \\
&- \frac{u^2v^2}{c^4}\sin^2\theta\ \sin^2\gamma\ \cos^2\delta + \frac{u^2v^2}{c^4}\sin^2\gamma\ \cos^2\delta + \frac{u^2v^2}{c^4}\cos^2\theta\ \sin^2\delta \\
&- \frac{v^2}{c^2}\cos^2\theta \\
&= \left(1 + \frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta\right)^2 - \frac{v^2}{c^2}\cos^2\theta\left(1 - \frac{u^2}{c^2}\sin^2\delta - \frac{u^2}{c^2}\sin^2\gamma\ \cos^2\delta\right) \\
&= \left(1 + \frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta\right)^2 \\
&- \frac{v^2}{c^2}\cos^2\theta\left(1 - \frac{u^2}{c^2} + \frac{u^2}{c^2}\cos^2\delta - \frac{u^2}{c^2}\cos^2\delta + \frac{u^2}{c^2}\cos^2\gamma\ \ \cos^2\delta\right) \\
&= \left(1 + \frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta\right)^2 - \frac{v^2}{c^2}\cos^2\theta\left(1 - \frac{u^2}{c^2}(1 - \cos^2\gamma\cos^2\delta)\right) \\
&= \left(1 + \frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta\right)^2 - \frac{v^2}{c^2}\cos^2\theta\left(1 - \frac{u^2}{c^2}\sin^2\alpha\right) \quad \text{(A9)}
\end{aligned}$$

From (A8) and (A9) we obtain the magnitude of the force (2) in the sphere model article Introduction:

$$F = |\mathbf{F}'| = \frac{q_1 q_2 \left(1 - \frac{u^2}{c^2}\right)\sqrt{\left(1 + \frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta\right)^2 - \frac{v^2}{c^2}\cos^2\theta\left(1 - \frac{u^2}{c^2}\sin^2\alpha\right)}}{4\pi\varepsilon_o r^2 \left(1 - \frac{u^2}{c^2}\ sin^2\ \alpha\right)^{\frac{3}{2}}\sqrt{1 - \frac{v^2}{c^2}}}$$

To obtain the direction in Σ of the force on q_2 as measured by a co-moving spring balance, we need to bear in mind that, as seen from Σ, the *x*-component of $\mathbf{F}'$ is length-contracted by the relativistic factor $\sqrt{1 - v^2/c^2}$. We thus obtain the direction vector (3) from the Introduction:

$$\begin{pmatrix} \cos\theta\left(1 - \frac{v^2}{c^2}\right) \\ \sin\theta + \frac{uv}{c^2}\sin\gamma\cos\delta \\ \frac{uv}{c^2}\sin\delta\cos\theta \end{pmatrix}$$

B Density factor calculations

I will determine the density factor $\overline{d_1}$ for a given constellation of two charges q_1 and q_2 moving at constant velocities **u** and **v** in Σ. Recall that, in Fig. 5 in the sphere model article, *A* is at the centre of the sphere on whose surface q_2 is located. Let the number of sphere surfaces that cut through $\overline{CB}$ be *m*. If q_1 took the time *t* to travel from *A* to *C*, we have

$$\overline{AB} = ct = \frac{m}{\lambda_\Sigma} \quad \text{(B1)}$$

and

$$\overline{AC} = ut \quad \text{(B2)}$$

Applying the cosine rule to the triangle *ABC* in Fig. 5, with $\alpha := \measuredangle(\mathbf{r}_{12}; \mathbf{u})$ as defined in Section 1, we thus have

$$u^2t^2 + \overline{CB}^2 + 2ut\overline{CB}\cos\alpha = c^2t^2 \quad \text{(B3)}$$

Solving for *t* and discarding the negative solution, we obtain

$$t = \frac{\overline{CB}}{c}\left(\sqrt{1 - \frac{u^2}{c^2}\sin^2\alpha} - \frac{u}{c}\cos\alpha\right)^{-1} \quad \text{(B4)}$$

Using (B1) and (B4), we have

$$\lambda_1^{<} = \frac{m}{\overline{CB}} = \frac{ct\lambda_\Sigma}{\overline{CB}} = \left(\sqrt{1 - \frac{u^2}{c^2}\sin^2\alpha} - \frac{u}{c}\cos\alpha\right)^{-1}\lambda_\Sigma \quad \text{(B5)}$$

In the same way, by applying the cosine rule to the triangle *ACD*, we obtain

$$\lambda_1^{>} = \left(\sqrt{1 - \frac{u^2}{c^2}\sin^2\alpha} + \frac{u}{c}\cos\alpha\right)^{-1}\lambda_\Sigma \quad \text{(B6)}$$

It turns out that the density on the near side and the far side of q_2 is independent of the number of sphere surfaces or the radius of the sphere surface on which q_2 is located. It is thus constant on the near side and the far side of q_2, respectively.

From (B5) and (B6), we finally obtain

$$\overline{d_1} = \sqrt{d_1^{<}\, d_1^{>}} = \sqrt{\frac{\lambda_1^{<}}{\lambda_\Sigma}\,\frac{\lambda_1^{>}}{\lambda_\Sigma}} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \quad \text{(B7)}$$

By applying the same procedure to q_2, we find that

$$\lambda_2^{<} = \left(\sqrt{1-\frac{v^2}{c^2}\sin^2\theta} + \frac{v}{c}\cos\theta\right)^{-1} \lambda_\Sigma \quad \text{(B8)}$$

$$\lambda_2^{>} = \left(\sqrt{1-\frac{v^2}{c^2}\sin^2\theta} - \frac{v}{c}\cos\theta\right)^{-1} \lambda_\Sigma \quad \text{(B9)}$$

$$\overline{d_2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{(B10)}$$

C Angle factor calculations

By applying the sine rule to the triangle *ABC* in Fig. 6, we obtain:

$$\frac{\overline{AB}}{\sin\alpha} = \frac{\overline{AC}}{\sin\left(\frac{\pi}{2}-\beta\right)} = \frac{\frac{u}{c}\,\overline{AB}}{\cos\beta} \quad \text{(C1)}$$

It follows that

$$\cos\beta = \frac{u}{c}\sin\alpha \quad \text{or} \quad \sin\beta = \sqrt{1-\frac{u^2}{c^2}\sin^2\alpha} \quad \text{(C2)}$$

The angle factor can now easily be calculated using (C2). Recall that it is the mean factor by which the trajectory taken by q_1 information from one sphere surface to the next in Fig. 6 is different from the local perpendicular distance between those sphere surfaces. By 'local perpendicular distance' I mean the distance between the tangent planes to those surfaces along the line $\overline{DB}$. This distance is well defined since those planes are all parallel to each other on the near and far side of q_1, respectively. It can be seen in Fig. 6 that the required factor is the same for the near side of q_1:

$$e_1^{<} = \frac{1}{\sin\beta} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}\sin^2\alpha}} \quad \text{(C3)}$$

as it is for the far side of q_1:

$$e_1^{>} = \frac{1}{\sin\beta} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}\sin^2\alpha}} \quad \text{(C4)}$$

Hence finally:

$$\bar{e}_1 = \sqrt{e_1^< e_1^>} = \frac{1}{\sin\beta} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}\sin^2\alpha}} \quad \text{(C6)}$$

D Frequency factor calculations

The frequency factor $\bar{f}_1$ introduced in the sphere model article depends on both the velocity **u** of q_1 and the velocity **v** of q_2. It is defined as the geometric mean of the frequency factors $f_1^<$ and $f_1^>$ associated with the near side and the far side of q_1.

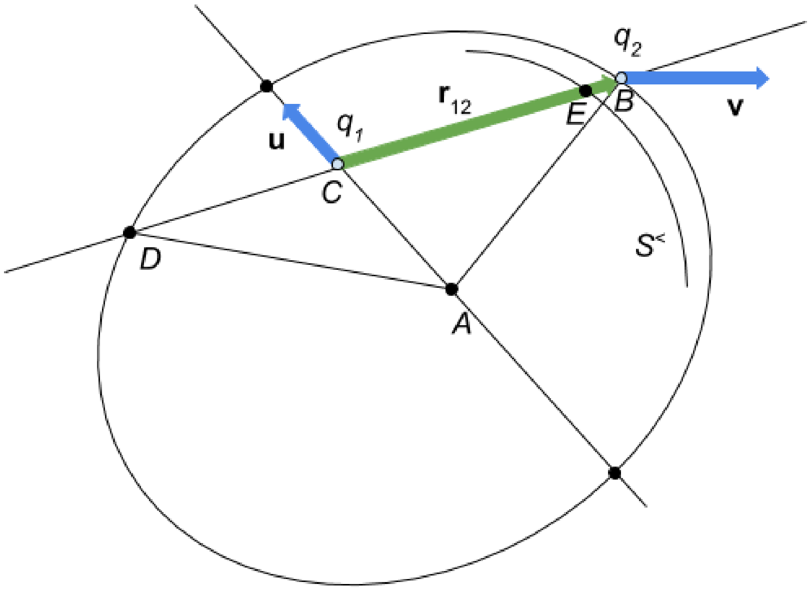


Fig. 10 *This diagram and those shown in subsequent figures are perspective drawings of the kind of three-dimensional situation shown in Fig. 1 in the sphere model article. Solid dots indicate points where lines meet or intersect. The point A is again at the centre of the q_1 sphere surface on which q_2 is located. To calculate the frequency factor $f_1^<$, we need to consider the q_1 information sphere surface $S^<$ that intersects the line connecting q_1 and q_2 in a point E very close to B. More precisely, $\overline{EB} = \Delta r_2^< = 1/\lambda_2^<$. The task is to determine how long it takes for $S^<$ and q_2 to meet.*

I will first calculate the frequency factor $f_1^<$. Recall that it is the factor by which the q_1 information transmission rate $\zeta_1^< = 1/t^<$ is different from the static case. I thus need to determine the time $t^<$ which it takes for the information sphere surface $S^<$ shown in Fig. 10 to reach q_2. In Fig. 10, $S^<$ intersects the line connecting q_1 and q_2 in a point E very close to B such that $\overline{EB} = \Delta r_2^< = 1/\lambda_2^< = \Delta r/d_2^< < 2\Delta r$.

Since the distance Δr between neighbouring sphere surfaces is assumed to be extremely small compared to the distance *r* between q_1 and q_2, $\overline{EB}$ is also extremely small compared to *r*. We can thus approximate $S^<$ in *E* by a plane $S_p^<$ characterized by the normal vector

$$\mathbf{n}_1^< = \frac{\mathbf{c}_1^<}{c} \quad \text{(D1)}$$

where $\mathbf{c}_1^<$ is the velocity of q_1 information arriving at q_2 from *A*, as shown in Fig. 11.

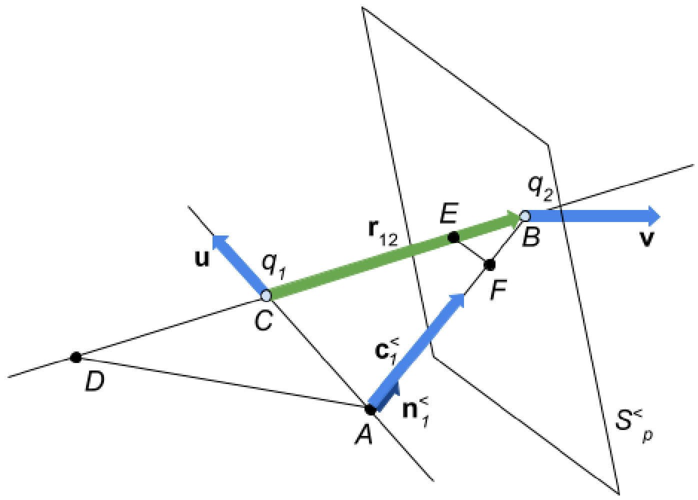


Fig. 11 *The line $\overline{EF}$ lies in the plane $S_p^<$, at a right angle to $\overline{FB}$, which is perpendicular to $S_p^<$ and represents the distance of $S_p^<$ from the origin of the coordinate system.*

The point F marks the intersection of $\overline{AB}$ with $S_p^<$, so $\overline{FB}$ is perpendicular to $S_p^<$ and represents the distance of $S_p^<$ from the origin of the coordinate system. The plane $S_p^<$, which moves in the direction of $\mathbf{n}_1^<$ at the speed c, can thus be written as follows:

$$\mathbf{x}\mathbf{n}_1^< + \overline{BF} - ct = 0 \quad \text{(D2)}$$

where $\mathbf{x}$ is a vector pointing to any point in $S_p^<$. To determine the time it takes for $S_p^<$ to meet q_2, we just need to insert $t = t^<$ and $\mathbf{x} = (vt^<, 0, 0)$ into (D2) and solve for $t^<$. Before we do so, we need to work out $\mathbf{n}_1^<$ and $\overline{BF}$. In Fig. 12

$$\mathbf{w}_1^< := \mathbf{c}_1^< - \mathbf{u} \quad \text{(D3)}$$

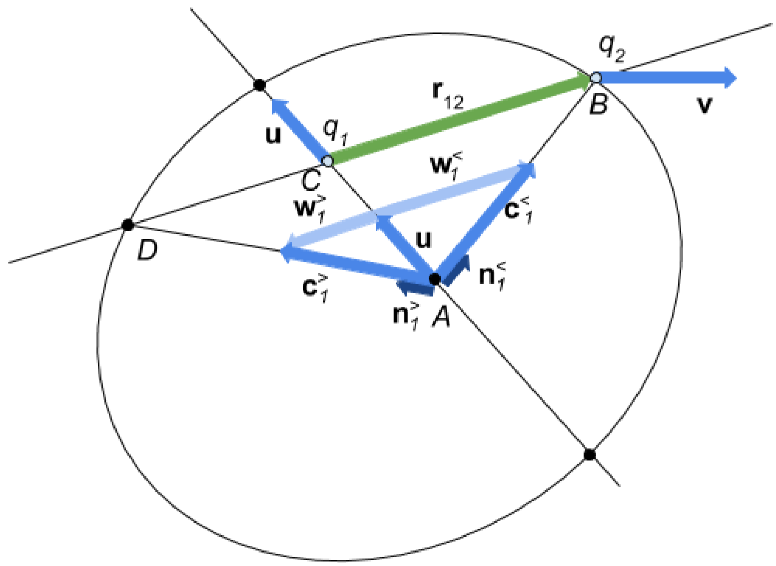


Fig. 12 *The vector $\mathbf{n}_1^<$ can be calculated by first determining $\mathbf{c}_1^<$, which can in turn be expressed in terms of $\mathbf{u}$ and $\mathbf{w}_1^<$. The magnitude and direction of $\mathbf{w}_1^<$ can be determined by means of elementary geometry.*

We can thus write

$$\mathbf{n}_1^< = \frac{\mathbf{c}_1^<}{c} = \frac{\mathbf{u} + \mathbf{w}_1^<}{c} \quad \text{(D4)}$$

We know that $\mathbf{w}_1^<$ is parallel to $\mathbf{r}_{12}$, so

$$\mathbf{w}_1^< = w_1^< \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \quad \text{(D5)}$$

Also in Fig. 12, we can see that

$$\frac{w_1^<}{\overline{CB}} = \frac{c}{\overline{AB}} \quad \text{(D6)}$$

Solving for $w_1^<$ and using (4), (B1), (B4) and (C2), we obtain

$$w_1^< = \frac{\overline{CB}c}{\overline{AB}} = c\left(\sqrt{1 - \frac{u^2}{c^2}\sin^2\alpha} - \frac{u}{c}\cos\alpha\right) = c\left(\sin\beta - \frac{u}{c}\cos\delta\cos\gamma\right) \quad \text{(D7)}$$

Using (A4), (D4), (D5) and (D7), we obtain:

$$\mathbf{n}_1^< = \frac{\mathbf{u} + \mathbf{w}_1^<}{c} = \begin{pmatrix} \frac{u}{c}(\cos\theta\cos\gamma - \sin\theta\sin\gamma)\cos\delta + \left(\sin\beta - \frac{u}{c}\cos\delta\cos\gamma\right)\cos\theta \\ \frac{u}{c}(\sin\theta\cos\gamma + \cos\theta\sin\gamma)\cos\delta + \left(\sin\beta - \frac{u}{c}\cos\delta\cos\gamma\right)\sin\theta \\ \frac{u}{c}\sin\delta \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{u}{c}\sin\theta\sin\gamma\cos\delta + \cos\theta\sin\beta \\ \frac{u}{c}\cos\theta\sin\gamma\cos\delta + \sin\theta\sin\beta \\ \frac{u}{c}\sin\delta \end{pmatrix} \quad \text{(D8)}$$

For $\overline{BF}$, we have

$$\overline{BF} = \sin\beta\,\overline{EB} = \sin\beta\,\Delta r_2^< = \frac{\sin\beta}{\lambda_2^<} \quad \text{(D9)}$$

We can now insert (D8) and (D9) into (D2). If we also set $t = t^<$ and $\mathbf{x} = (vt^<, 0, 0)$ in (D2), we obtain:

$$vt^<\left(-\frac{u}{c}\sin\theta\sin\gamma\cos\delta + \cos\theta\sin\beta\right) + \frac{\sin\beta}{\lambda_2^<} - ct^< = 0 \quad \text{(D10)}$$

Rearranging, we have

$$ct^< + vt^<\left(\frac{u}{c}\sin\theta\sin\gamma\cos\delta - \cos\theta\sin\beta\right) = \frac{\sin\beta}{\lambda_2^<} \quad \text{(D11)}$$

Solving for $t^<$ and using (B8) for $\lambda_2^<$, we obtain:

$$t^{<}=\frac{\Delta r}{c}\frac{\sin\beta\left(\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}+\frac{v}{c}\cos\theta\right)}{1+\frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta-\frac{v}{c}\cos\theta\sin\beta}\quad \text{(D12)}$$

Now $\zeta_1^{<}=1/t^{<}$ is the q_1 information transmission rate and $\zeta_\Sigma=c/\Delta r$ is the same rate in the static case. We thus obtain

$$f_1^{<}=\frac{\zeta_1^{<}}{\zeta_\Sigma}=\frac{1+\frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta-\frac{v}{c}\cos\theta\sin\beta}{\sin\beta\left(\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}+\frac{v}{c}\cos\theta\right)}\quad \text{(D13)}$$

We can use the same method to obtain $\mathbf{n}_1^{>}$ and $f_1^{>}$. However, in this case we have to define $S_p^{>}$ as a plane which approaches q_2 at c from the far side of q_2, as shown in Fig. 13.

In terms of the sphere model, this does not mean that q_1 information is approaching B from some distant point on the far side of q_2. Rather, it means that the information arriving at B from q_1 includes full information not just about $\mathbf{n}_1^{<}$ but also about $\mathbf{n}_1^{>}$.

The plane equation is now

$$\mathbf{x}\mathbf{n}_1^{>}+\overline{BH}-ct=0\quad \text{(D14)}$$

$\overline{BG}=\Delta r_2^{>}$, so

$$\overline{BH}=\sin\beta\,\overline{BG}=\sin\beta\,\Delta r_2^{>}=\frac{\sin\beta}{\lambda_2^{>}}\quad \text{(D15)}$$

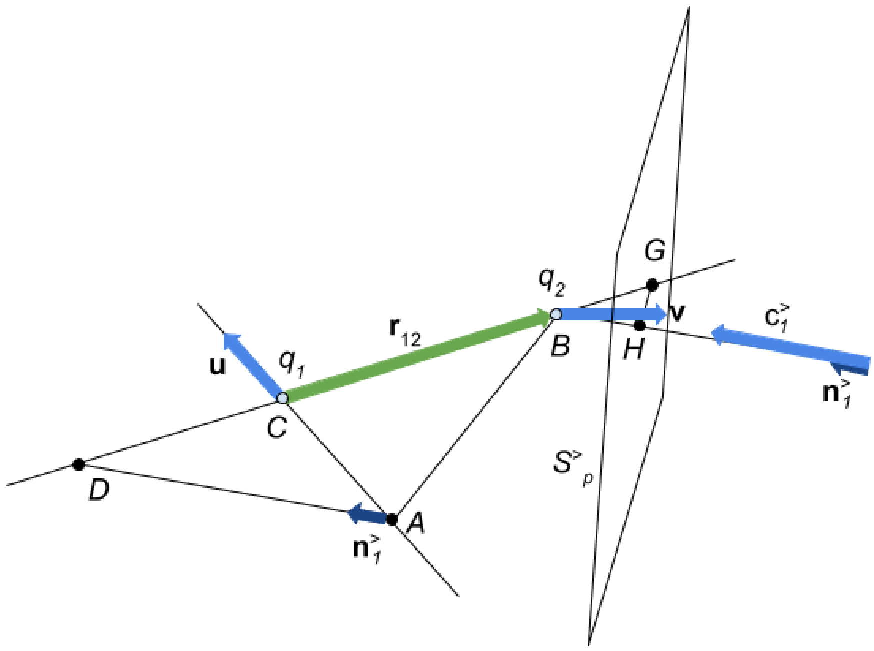


Fig. 13 *To obtain* $\mathbf{n}_1^{>}$ *and* $f_1^{>}$, *we have to define* $S_p^{>}$ *as a plane which approaches* q_2 *at c from the far side of* q_2.

We can work out $\mathbf{n}_1^{>}$ in the same way as $\mathbf{n}_1^{<}$, using $\mathbf{w}_1^{>}$ in Fig. 12 instead of $\mathbf{w}_1^{<}$. We obtain

$$\mathbf{n}_1^{>} = \begin{pmatrix} -\frac{u}{c}\sin\theta\sin\gamma\cos\delta - \cos\theta\sin\beta \\ \frac{u}{c}\cos\theta\sin\gamma\cos\delta - \sin\theta\sin\beta \\ \frac{u}{c}\sin\delta \end{pmatrix} \quad \text{(D16)}$$

Proceeding in the same way as for $f_1^{<}$, we obtain

$$f_1^{>} = \frac{\zeta_1^{>}}{\zeta_\Sigma} = \frac{1+\frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta + \frac{v}{c}\cos\theta\sin\beta}{\sin\beta\left(\sqrt{1-\frac{v^2}{c^2}\sin^2\theta} - \frac{v}{c}\cos\theta\right)} \quad \text{(D17)}$$

and thus finally

$$\bar{f}_1 = \sqrt{f_1^{<}\, f_1^{>}} = \frac{\sqrt{\left(1+\frac{uv}{c^2}\sin\theta\sin\gamma\cos\delta\right)^2 - \frac{v^2}{c^2}\cos^2\theta\,\sin^2\beta}}{\sin\beta\sqrt{1-\frac{v^2}{c^2}}} \quad \text{(D18)}$$

E The direction of the force

I need to show that

$$\mathbf{a} := e_2^{<}\left(\widehat{\mathbf{n}_1^{<} - \frac{\mathbf{v}}{c}}\right) - e_2^{>}\left(\widehat{\mathbf{n}_1^{>} - \frac{\mathbf{v}}{c}}\right) \parallel \begin{pmatrix} \cos\theta\left(1-\frac{v^2}{c^2}\right) \\ \sin\theta + \frac{uv}{c^2}\sin\gamma\cos\delta \\ \frac{uv}{c^2}\sin\delta\cos\theta \end{pmatrix} \quad \text{(E1)}$$

We know that

$$e_2^{<} = 1/\cos\sphericalangle(\mathbf{n}_1^{<}; \mathbf{n}_1^{<} - \mathbf{v}/c) = \frac{|\mathbf{n}_1^{<}|\left|\mathbf{n}_1^{<} - \frac{\mathbf{v}}{c}\right|}{\mathbf{n}_1^{<}\left(\mathbf{n}_1^{<} - \frac{\mathbf{v}}{c}\right)} = \frac{\left|\mathbf{n}_1^{<} - \frac{\mathbf{v}}{c}\right|}{1 - n_{1x}^{<}\frac{v}{c}} \quad \text{(E2)}$$

and

$$e_2^{>} = 1/\cos\sphericalangle(\mathbf{n}_1^{>}; \mathbf{n}_1^{>} - \mathbf{v}/c) = \frac{|\mathbf{n}_1^{>}|\left|\mathbf{n}_1^{>} - \frac{\mathbf{v}}{c}\right|}{\mathbf{n}_1^{>}\left(\mathbf{n}_1^{>} - \frac{\mathbf{v}}{c}\right)} = \frac{\left|\mathbf{n}_1^{>} - \frac{\mathbf{v}}{c}\right|}{1 - n_{1x}^{>}\frac{v}{c}} \quad \text{(E3)}$$

Hence

$$\mathbf{a} = \frac{1}{1-n_{1x}^{<}\frac{v}{c}}\left(\mathbf{n}_1^{<} - \frac{\mathbf{v}}{c}\right) - \frac{1}{1-n_{1x}^{>}\frac{v}{c}}\left(\mathbf{n}_1^{>} - \frac{\mathbf{v}}{c}\right) \quad \text{(E4)}$$

We can define a vector **b** that is parallel to **a** as follows:

$$\mathbf{b} := \left(1-n_{1x}^{<}\frac{v}{c}\right)\left(1-n_{1x}^{>}\frac{v}{c}\right)\mathbf{a} = \left(1-n_{1x}^{>}\frac{v}{c}\right)\left(\mathbf{n}_1^{<} - \frac{\mathbf{v}}{c}\right) - \left(1-n_{1x}^{<}\frac{v}{c}\right)\left(\mathbf{n}_1^{>} - \frac{\mathbf{v}}{c}\right) \quad \text{(E5)}$$

I will show (E1) by proving that

$$\mathbf{b} \parallel \begin{pmatrix} \cos\theta\left(1-\frac{v^2}{c^2}\right) \\ \sin\theta + \frac{uv}{c^2}\sin\gamma\cos\delta \\ \frac{uv}{c^2}\sin\delta\cos\theta \end{pmatrix} \quad \text{(E6)}$$

For brevity, it is convenient to introduce the following abbreviations:

$$k_1 = \frac{u}{c}\sin\theta\sin\gamma\cos\delta \quad \text{(E7)}$$

$$k_2 = \frac{u}{c}\cos\theta\sin\gamma\cos\delta \quad \text{(E8)}$$

$$k_3 = 1 + k_1\frac{v}{c} = 1 + \frac{uv}{c}\sin\theta\sin\gamma\cos\delta \quad \text{(E9)}$$

Bearing in mind (D8) and (D16), we can then write

$$\mathbf{n}_1^{<} = \begin{pmatrix} -k_1 + \cos\theta\sin\beta \\ k_2 + \sin\theta\sin\beta \\ \frac{u}{c}\sin\delta \end{pmatrix} \quad \text{(E10)}$$

and

$$\mathbf{n}_1^{>} = \begin{pmatrix} -k_1 - \cos\theta\sin\beta \\ k_2 - \sin\theta\sin\beta \\ \frac{u}{c}\sin\delta \end{pmatrix} \quad \text{(E11)}$$

We thus have for the components of **b**:

$$\begin{aligned} b_x = &\left(k_3 + \frac{v}{c}\cos\theta\sin\beta\right)\left(-k_1 + \cos\theta\sin\beta - \frac{v}{c}\right) \\ &- \left(k_3 - \frac{v}{c}\cos\theta\sin\beta\right)\left(-k_1 - \cos\theta\sin\beta - \frac{v}{c}\right) \\ &= 2\cos\theta\sin\beta\left(k_3 - \frac{v}{c}k_1 - \frac{v^2}{c^2}\right) \\ &= 2\cos\theta\sin\beta\left(1 + \frac{uv}{c}\sin\theta\sin\gamma\cos\delta - \frac{uv}{c}\sin\theta\sin\gamma\cos\delta - \frac{v^2}{c^2}\right) \\ &= 2\sin\beta\cos\theta\left(1 - \frac{v^2}{c^2}\right) \quad \text{(E12)} \end{aligned}$$

$$b_y = \left(k_3 + \frac{v}{c}\cos\theta\sin\beta\right)(k_2 + \sin\theta\sin\beta) - \left(k_3 - \frac{v}{c}\cos\theta\sin\beta\right)(k_2 - \sin\theta\sin\beta)$$
$$= 2\sin\beta\left(k_3\sin\theta - \frac{v}{c}k_2\cos\theta\right)$$
$$= 2\sin\beta\left(\sin\theta + \frac{uv}{c}\sin^2\theta\sin\gamma\cos\delta + \frac{uv}{c}\cos^2\theta\sin\gamma\cos\delta\right)$$
$$= 2\sin\beta\left(\sin\theta + \frac{uv}{c}\sin\gamma\cos\delta\right) \quad \text{(E13)}$$

$$b_z = \left(k_3 + \frac{v}{c}\cos\theta\sin\beta\right)\frac{u}{c}\sin\delta - \left(k_3 - \frac{v}{c}\cos\theta\sin\beta\right)\frac{u}{c}\sin\delta$$
$$= 2\sin\beta\frac{uv}{c}\sin\delta\cos\theta \quad \text{(E14)}$$

Collecting the results, we obtain

$$\mathbf{b} = 2\sin\beta\begin{pmatrix}\cos\theta\left(1 - \frac{v^2}{c^2}\right)\\ \sin\theta + \frac{uv}{c^2}\sin\gamma\cos\delta\\ \frac{uv}{c^2}\sin\delta\cos\theta\end{pmatrix} \quad \text{(E15)}$$

We have thus proved (E6) and, as a result, also (E1).

F From length contraction to time dilation and the constancy of *c*

a) From length contraction in Σ to time dilation in Σ.

I would like to show that a light clock of any orientation moving at **v** through an inertial frame of reference Σ in which light propagates in isotropic conditions and clocks have been Einstein-adjusted ticks more slowly by the relativistic factor $\sqrt{1 - v^2/c^2}$, as measured in Σ, than a light clock of identical construction which is stationary in Σ.

I will assume that in Σ light propagates at the speed *c* in all directions, and that objects moving through Σ are shortened in the direction of movement by the relativistic factor $\sqrt{1 - v^2/c^2}$. I will not assume anything about the speed of light in the moving system S.

Fig. 16 shows a light clock represented by the triangle $P_2P_3P_5$ moving at **v** through Σ:

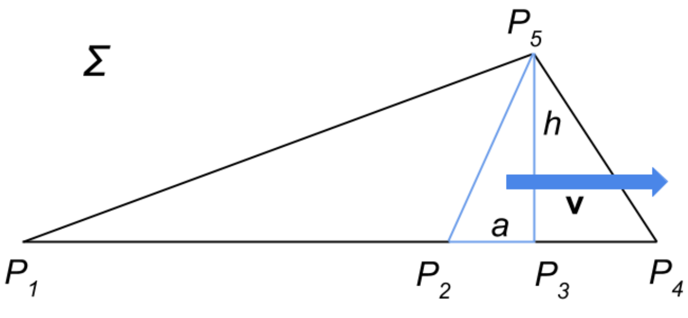


Fig. 16 *The moving light clock is represented by the triangle* $P_2P_3P_5$.

The light is sent from P_2 to P_5 and back, so in Σ the light clock is inclined by an angle which is defined by a and the light clock height h.

The light is sent out as P_2 passes P_1, and it arrives in P_5 at the time t_1, which is the moment shown in the diagram. It is then reflected and arrives back in P_2 after the time t_2, as P_2 reaches P_4.

We thus have $\overline{P_1P_2} = vt_1$; $\overline{P_1P_5} = ct_1$; $\overline{P_2P_4} = vt_2$; and $\overline{P_5P_4} = ct_2$ and therefore:

$$(vt_1 \pm a)^2 + h^2 = c^2t_1^2 \quad \text{(F1)}$$

and

$$(vt_2 \mp a)^2 + h^2 = c^2t_2^2 \quad \text{(F2)}$$

The combination of plus and minus signs depends on whether the light clock is oriented to the right or to the left in Fig. 16. From (F1) and (F2), it is possible to calculate the period T_S of the moving light clock as measured in Σ:

$$T_S = t_1 + t_2 = \frac{2c\sqrt{a^2 + h^2\left(1 - \frac{v^2}{c^2}\right)}}{c^2 - v^2} \quad \text{(F3)}$$

Next we need to calculate the period T_Σ of the light clock $P_2P_3P_5$ after it has been brought to a halt in Σ. Its previously contracted base a is then extended by the relativistic factor to become

$$\frac{a}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{(F4)}$$

Moreover, P_2P_5 is now equal to $cT_\Sigma/2$, so we have:

$$\left(\frac{cT_\Sigma}{2}\right)^2 = h^2 + \frac{a^2}{1 - \frac{v^2}{c^2}} \quad \text{(F5)}$$

It follows that

$$T_\Sigma = 2\sqrt{\frac{h^2\left(1 - \frac{v^2}{c^2}\right) + a^2}{c^2 - v^2}} \quad \text{(F6)}$$

and thus

$$\frac{T_\Sigma}{T_S} = \frac{c^2 - v^2}{c\sqrt{c^2 - v^2}} = \sqrt{1 - \frac{v^2}{c^2}} \quad \text{(F7)}$$

which is what I set out to prove.

If we assume that other kinds of clock behave in the same way as light clocks, then it follows that all clocks moving through Σ run slow by the relativistic factor compared to stationary clocks in Σ which they pass.

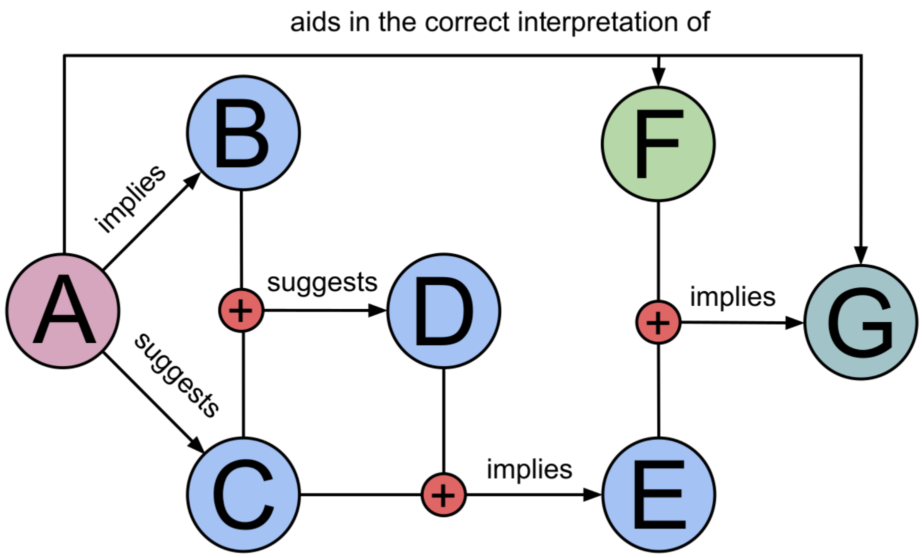
b) From length contraction and time dilation in Σ to the constancy of the two-way speed of light in any inertial frame of reference.

To calculate the two-way speed of light in the moving frame, we need to calculate the time *t'* it takes for light to travel from P_2 to P_5 and back in Fig. 16 as measured by a clock at rest in S at P_2, and the distance *l'* from P_2 to P_5 and back as measured in S. Using (F3) and bearing in mind length contraction, we obtain for the two-way speed of light *c'* as measured in *S*:

$$c' = \frac{l'}{t'} = \frac{2\sqrt{h^2 + \dfrac{a^2}{1-\dfrac{v^2}{c^2}}}}{T_S\sqrt{1-\dfrac{v^2}{c^2}}} = \frac{2\sqrt{\dfrac{h^2\left(1-\dfrac{v^2}{c^2}\right)+a^2}{1-\dfrac{v^2}{c^2}}}}{\dfrac{2c\sqrt{a^2+h^2\left(1-\dfrac{v^2}{c^2}\right)}}{c^2-v^2}\sqrt{1-\dfrac{v^2}{c^2}}} = c \quad \text{(F8)}$$

This completes the proof. The constancy of the one-way speed of light is a consequence of Einstein's clock adjustment procedure, which ensures that light always travels at the same speed in opposite directions. 'Speed' means a purely formal 'coordinate speed' here since locally Einstein-adjusted clocks in any system S moving relative to Σ are not synchronized.

The various dependencies are shown schematically below:



A = The sphere model of electricity
B = Independence of the speed of light from the speed of the source
C = Length contraction in a single inertial frame of reference
D = Time dilation in the same frame of reference
E = Constancy of the two-way speed of light in any frame of reference
F = Einstein's clock adjustment convention
G = Constancy of the one-way speed of light